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An explicit formula for Koecher–Maaß Dirichlet series for Eisenstein series of Klingen type

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Abstract

We give a reasonable expression of the Koecher–Maaß Dirichlet series for the Klingen–Eisenstein lift of an elliptic cusp form.

1. Introduction

Let $f(Z)$ be a Siegel modular form of weight k belonging to the symplectic group $\Gamma_n = Sp_n(\mathbf{Z})$. Then $f(Z)$ has the following Fourier expansion:

$$f(Z) = \sum_A c_f(A) \exp(2\pi i \operatorname{tr}(AZ)),$$

where A runs over all semi-positive definite half-integral matrices over $\mathbf{Z}$ of degree n and $\operatorname{tr}(X)$ denotes the trace of a matrix X. We then define the Koecher–Maaß Dirichlet series $L(f,s)$ by

$$L(f,s) = \sum_A \frac{c_f(A)}{r(A,A)(\det A)^s},$$

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where A runs over a complete set of representatives of $GL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n, and $r(A, A)$ denotes the order of the orthogonal group of A. The Koecher–Maaß Dirichlet series can also be obtained as the Mellin transform of f, and therefore its analytic properties are relatively known. As for this, we refer to [Ar1,Ar2,M]. However we had little knowledge about its arithmetic properties. Thus we present the following problem:

Problem 1. Investigate the arithmetic properties of $L(f, s)$.

To this problem, Böcherer and Schulze-Pillot have made a large contribution. As for this, we refer to [B-S1,B-S2,B-S3]. In those papers, they mainly treat the case of Yoshida lifting. In this paper, we take another approach to this problem. Namely we consider the Koecher–Maaß Dirichlet series for Eisenstein series of Klingen type; let f be a cusp form of weight k belonging to Γ_r $(0 \leqslant r \leqslant n \leqslant k - r - 2,)$ and define $[f]_r^n(Z)$ as

$$[f]_r^n(Z) = \sum_{M \in \Delta_{n,r} \backslash \Gamma_n} f(M\langle Z\rangle^*) j(M, Z)^{-k},$$

where $\Delta_{n,r} = \{ \begin{pmatrix} * & * \\ O_{n-r,n+r} & * \end{pmatrix} \in \Gamma_n \}$, and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ let $M\langle Z\rangle^*$ denote the upper left $(r \times r)$-block of the matrix $(AZ + B)(CZ + D)^{-1}$ and $j(M, Z) = \det(CZ + D)$. We note that $[1]_0^n(Z)$ is nothing but the Siegel Eisenstein series $E_{n,k}(Z)$ of weight k. We then propose the following problem:

Problem 2. Let $0 \leqslant r < n$. Then give an explicit form of $L([f]_r^n, s)$ in terms of f.

In [B2], among others, Böcherer gave an explicit form of $L([f]_1^2, s)$ and $L(E_{2,k}, s)$. In [I-K1] we gave an explicit form of $L(E_{n,k}, s)$ for arbitrary n. We note that $L(E_{n,k}, s)$ is also regarded as the zeta function of prehomogeneous vector space. From this point of view, Saito gave a generalization of our result (cf. [Sa]). In relation to Problem 2 we should add one remark; in the explicit formula for $L([f]_1^2, s)$ by [B2], a certain Dirichlet series attached to f appears. Böcherer obtained a functional equation for it from the general theory of the Koecher–Maaß Dirichlet series. This Dirichlet series is a modification of the Dirichlet series originally defined by Kohnen and Zagier [K–Z], and is of importance in its own right. Hence the following problem seems very interesting.

Problem 3. Investigate the analytic and arithmetic properties of the Dirichlet series related to f appearing in an explicit formula for $L([f]_r^n, s)$.

In this paper, we exclusively treat the case of $[f]_1^n$ with f a cusp form belonging to Γ_1 and n even, and give an answer to Problems 2 and 3. This also gives a certain generalization of Böcherer's result in [B2].

Now to state our main result, for the fundamental discriminant d of a quadratic field, let ψ_d denote the Kronecker character associated with d. Here we understand that $\psi_1 = 1$. For $l = \pm 1$, put

$$\mathscr{F}_l = \{D_0 \in \mathbf{Z}_{>0}; lD_0 \text{ is the fundamental discriminant of a quadratic field or } 1\}.$$

For an integer D such that $lD > 0$ and $D \equiv 1$ or $\equiv 0 \bmod 4$, write $D = lD_0 m^2$ with $D_0 \in \mathscr{F}_l, m > 0$, and put

$$L_D(s) = L(s, \psi_{lD_0}) \sum_{d|m} \mu(d)\psi_{lD_0}(d)d^{-s} \sum_{c|md^{-1}} c^{1-2s},$$

where $L(s, \psi_{lD_0})$ is the Dirichlet L-function attached to ψ_{lD_0}, and μ is the Möbius function. Write $L_D(s)$ as

$$L_D(s) = \sum_{e=1}^{\infty} \epsilon_D(e)e^{-s},$$

and for a cusp form $f(z) = \sum_{e=1}^{\infty} b(e)\exp(2\pi i e z)$ of weight k with respect to $SL_2(\mathbf{Z})$ put

$$L(f, D, s) = \sum_{e=1}^{\infty} \epsilon_D(e)b(e)e^{-s},$$

and in particular put

$$\zeta(f, s) = L(f, 1, s).$$

We note that

$$\zeta(f, s) = 2L(f, s).$$

Further for $l = \pm 1$

$$\mathscr{L}_l(f; \lambda, s) = \sum_D L(f, lD, \lambda)D^{-s},$$

where D runs over all positive integers such that $D \equiv l, 0 \bmod 4$. This type of Dirichlet series was originally introduced by Kohnen and Zagier [K–Z]. Assume that f is a Hecke eigenform. Then we note that

$$\begin{aligned}\mathscr{L}_l(f; \lambda, s) = & \frac{\zeta^{\mathrm{st}}(f, 2s + 2\lambda - k)\zeta(2s)}{\zeta(2s + 2\lambda - k)} \sum_{D_0 \in \mathscr{F}_l} D_0^{-s} L(f, lD_0, \lambda) \\ & \times \prod_p \{(1 + \psi_{lD_0}(p)^2 p^{-2s+k-1-2\lambda})(1 + p^{-2s+k-2\lambda}) \\ & - \psi_{lD_0}(p)b(p)p^{-2s-\lambda}(1 + p^{k-2\lambda})\},\end{aligned}$$

where $\zeta(s)$ is Riemann's zeta function and $\zeta^{\mathrm{st}}(f,s)$ is the standard zeta function of f, which will be defined in the next section (cf. [B1]). Now for integers n and k put

$$\beta_{n,k} = (-1)^{k/2}\varepsilon_n \pi^{(n-1)k-n^2/2+n/2}2^{-k+n/2-1}$$
$$\times \prod_{i=1}^{n-1} \frac{\Gamma(i/2)}{\Gamma(k-i/2)} \prod_{i=1}^{n/2-1} \frac{\zeta(2i)}{\zeta(2k-2i-2)},$$

where $\varepsilon_n = 1/2$ or 1 according as $n=2$ or not.

Theorem 1. *Let n be an even positive integer. Then, under the above assumption, we have*

$$\begin{aligned} L([f]_1^n,s) = 2^{ns}\beta_{n,k}\Bigg[&\frac{\zeta(f,k-n/2)}{\zeta^{\mathrm{st}}(f,k-1)}\zeta(2s-1)\prod_{i=1}^{n/2-1}\zeta(2s-2i-1)\zeta(2s-2k+2i+2)\\ &\times\mathscr{L}_{(-1)^{n/2}}(f;k-1,s-k+3/2)\\ &+(-1)^{n(n-2)/8}\frac{\zeta(f,k-1)}{\zeta^{\mathrm{st}}(f,k-1)}\zeta(2s-n+1)\\ &\times\prod_{i=1}^{n/2-1}\zeta(2s-2i)\zeta(2s-2k+2i+1)\\ &\times\mathscr{L}_{(-1)^{n/2}}(f;k-n/2,s-k+(n+1)/2)\Bigg].\end{aligned}$$

By the above theorem combined with the general theory of $L([f]_1^n,s)$ obtained by Maaß [M], we obtain

Corollary. *Assume that* $n \equiv 2 \bmod 4$. *Put*

$$\mathbf{L}_{-1}(f;\lambda,s) = \pi^{(2\lambda-2k)(s+\lambda-1/2)}\zeta(2s+4\lambda-2k)\Gamma(s+\lambda-1/2)\Gamma(s+\lambda-1)\mathscr{L}_{-1}(f;\lambda,s).$$

Then $\mathbf{L}_{-1}(f;k-n/2,s)$ *can be continued analytically to a meromorphic function of s in the whole complex plane, and has the following functional equation*:

$$\mathbf{L}_{-1}(f;k-n/2,n+1-s-k) = \mathbf{L}_{-1}(f;k-n/2,s).$$

Remark. If $n=2$, the two terms in the above formula coincide with each other, and unify in one term. This is nothing but Böcherer's result [B2,Satz 2].

Theorem 1 cannot be derived directly from the commutativity of the Siegel operator and Hecke operators. The main idea of the proof is to relate the Koecher–Maaß Dirichlet series for a modular form to the standard zeta function for it. The present paper is organized as follows; in Section 2, we review the result in [I-K2] concerning the expression of the Koecher–Maaß Dirichlet series $L(F,s)$ for a general Siegel modular form F in terms of the standard zeta function and the *primitive Fourier coefficients* of F. (cf. Theorem 2). In Section 3, applying Theorem 2 to $F=[f]_1^n$ with f a cusp form belonging to Γ_1, we express $L([f]_1^n,s)$ as a sum of certain Euler products (cf. Theorem 3.2). After giving several preliminary results in Section 4, we complete the proof in the final section.

Notation. For a real number r we denote by $[r]$ the greatest integer not exceeding r. For a subset S of a commutative ring R, put $S^{\square}=\{a^2; a\in S\}$. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m,n)-matrices with entries in R. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m=0$ or $n=0$. In particular put $M_n(R)=M_{nn}(R)$. For an (m,n)-matrix X and an (m,m)-matrix A, we write $A[X]={}^tXAX$, where ${}^tX$ denotes the transpose of X. We denote by E_n and by $O_{n,r}$ the unit matrix of degree n and the (n,r)-zero matrix, respectively. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset $X \bmod aM_{mn}(R)$. Put

$$GL_m(R)=\{A\in M_m(R); \det A\in R^*\},$$

where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathscr{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathscr{H}_n(R)$ is the set of symmetric matrices of degree n whose (i,j)-component belongs to R or $\frac{1}{2}R$ according as $i=j$ or not. We define the set $\mathscr{E}_n(R)$ of even-integral matrices over R by $\mathscr{E}_n(R)=2\mathscr{H}_n(R)$. For a subset S of $M_n(R)$ we denote by $S^{\times}$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geqslant 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R' with each other and write $A\underset{R'}{\sim}A'$ if there is an element X of $GL_n(R')$ such that $A'=A[X]$. We also write $A\sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X\perp Y=\begin{pmatrix} X & O\\ O & Y\end{pmatrix}$.

Let (M,q) be a quadratic module over a ring R with a quadratic form q in the sense of [Ki2], and b the associated symmetric bilinear form defined by $b(x,y)=q(x+y)-q(x)-q(y)$ for $x,y\in M$. In particular, if R is an integral domain of characteristic different from 2, we define a bilinear form $\tilde{b}$ on M by $\tilde{b}(x,y)=\frac{1}{2}b(x,y)$. We define submodules $M^{\perp}$ and $\mathrm{Rad}\, M$ by

$$M^{\perp}=\{x\in M; b(x,y)=0 \text{ for any } y\in M,\}$$

and

$$\mathrm{Rad}\, M=\{x\in M^{\perp}; q(x)=0 \text{ for any } x\in M\}.$$

We note that $M^{\perp}=\mathrm{Rad}\, M$ if the characteristic of R is not 2. We say that q is non-degenerate if $\mathrm{Rad}\, M=\{0\}$. For a half-integral matrix A over a ring R of degree n, let q_A be the quadratic form of $M_{n1}(R)$ over R defined by $q_A(\mathbf{x}) = A[\mathbf{x}]$ for $\mathbf{x} \in M_{n1}(R)$. Then the quadratic space $M_A = (M_{n1}(R), q_A)$ is called the quadratic space associated with A.

Let F be a field of characteristic different from 2, and R a subring of F. Let (V, q) be a quadratic space over F, and L a quadratic R-lattice in V, that is, a finitely generated R-module such that $L \otimes_R F = V$. For a symmetric matrix A with entries in F, let $M_A = (M_{n1}(F), q_A)$ be the quadratic space associated with A as above. Then $M_{n1}(R)$ can be regarded as a quadratic R-lattice of M_A in a natural manner, which we write as L_A.

Let G be a group, and Y a right (resp. left) G-set. Then we denote by Y/G (resp. $G\backslash Y$) the set of right (resp. left) equivalence classes of Y under G.

2. Koecher–Maaß Dirichlet series and the standard zeta function

In this section, we review an expression of the standard zeta function of a Siegel modular form in terms of the squared Möbius function. As for the details, see [I-K2].

A half-integral matrix A over $\mathbf{Z}_p$ is called non-degenerate modulo p if the reduction $M_A \otimes_{\mathbf{Z}_p} \mathbf{Z}_p/p\mathbf{Z}_p$ of the quadratic space M_A is non-degenerate. Define a subset $\mathscr{K}_n{}'(\mathbf{Z}_p)$ of $\mathscr{H}_n(\mathbf{Z}_p)$ by

$$\mathscr{K}_n{}'(\mathbf{Z}_p) = \{A \in \mathscr{H}_n(\mathbf{Z}_p); A \sim V_0 \perp pV_1 \text{with } V_0, V_1 \text{ non-degenerate modulo } p\}.$$

Further define a subset $\mathscr{K}''_n(\mathbf{Z}_2)$ of $\mathscr{H}_n(\mathbf{Z}_2)$ by

$$\begin{aligned}\mathscr{K}''_n(\mathbf{Z}_2) = \{A \in \mathscr{H}_n(\mathbf{Z}_2); A \sim \tfrac{1}{2}V_0 \perp V \perp V_1 \text{ with } V_0, V_1 \text{ even-integral unimodular}\\ \text{and } V \text{ a diagonal unimodular matrix of degree 2 such}\\ \text{that } \det V \equiv 1 \bmod 4\}.\end{aligned}$$

Put $\mathscr{K}_n(\mathbf{Z}_p) = \mathscr{K}_n{}'(\mathbf{Z}_2) \cup \mathscr{K}''_n(\mathbf{Z}_2)$ or $\mathscr{K}_n{}'(\mathbf{Z}_p)$ according as $p = 2$ or not. For a p-adic number c put

$$\chi_p(c) = 1, -1 \text{ or } 0$$

according as $\mathbf{Q}_p(\sqrt{c}) = \mathbf{Q}_p, \mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic unramified, or $\mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic ramified. Further for a symmetric matrix A of even degree n with entries in $\mathbf{Q}_p$ we put

$$\xi_p(A) = \chi_p((-1)^{n/2} \det A).$$

For a non-degenerate half-integral matrix A we define $\sigma_p(A)$ as follows; first assume that A belongs to $\mathscr{K}_n{}'(\mathbf{Z}_p)$. Then we have $A \sim V_0 \perp pV_1$ with V_0, V_1 non-degenerate matrices modulo p of degree n_0 and n_1, respectively. Then we put

$$\sigma_p(A) = \begin{cases} (-1)^{n_1/2}\xi_p(V_1)p^{(n_1^2-2n_1)/4} & \text{if } n_1 \text{ is even}, \\ (-1)^{(n_1-1)/2}p^{(n_1-1)^2/4} & \text{if } n_1 \text{ is odd}. \end{cases}$$

Next let $p=2$ and assume that A belongs to $\mathscr{K}''_n(\mathbf{Z}_2)$. Then we have $A \sim \frac{1}{2}V_0 \perp V \perp V_1$ with V_0, V_1 even-integral unimodular matrices of degree n_0 and n_1, respectively, and V a unimodular diagonal matrix of degree 2 such that $\det V \equiv 1 \bmod 4$. Then we put

$$\sigma_p(A) = (-1)^{n_1/2}p^{n_1^2/4}.$$

Finally if A does not belong to $\mathscr{K}_n(\mathbf{Z}_p)$ we put $\sigma_p(A)=0$. For a non-degenerate half-integral matrix A over $\mathbf{Z}$ put

$$\sigma(A) = \prod_p \sigma_p(A).$$

By definition $\sigma(A)$ depends only on the genus of A. Put $\mathscr{K}_n(\mathbf{Z}) = \mathscr{H}_n(\mathbf{Z})^\times \cap \prod_p \mathscr{K}_n(\mathbf{Z}_p)$. Then by definition we have $\sigma(A)=0$ for $A \notin \mathscr{K}_n(\mathbf{Z})$. We remark that $\mathscr{H}_1(\mathbf{Z})^\times$ can be identified with the set of all non-zero integers. Further, by definition we have $\sigma(a)=1$ or 0 according as a is square free or not, and therefore, σ is nothing but the square of the usual Möbius function in case $n=1$. Thus we call σ the squared Möbius function over $\mathscr{H}_n(\mathbf{Z})$. Now for a non-degenerate positive definite half-integral matrices A and C of degree n over $\mathbf{Z}$ put

$$G(A,C) = \sum_{A' \in \mathscr{G}(A)} \frac{r(A',C)}{r(A',A')},$$

where $\mathscr{G}(A)$ denotes the set of equivalence classes belonging to the genus of A, and $r(A,C)$ the representation number of C by A. As is well-known $G(A,C)$ is determined by $\mathscr{G}(A)$ and $\mathscr{G}(C)$.

From now on for a p-adic number c let $v(c)=v_p(c)$ denote the normalized additive valuation on $\mathbf{Q}_p$. Now for a half-integral matrix A over $\mathbf{Z}_p$ put $\bar{M}_A = M_A \otimes \mathbf{Z}_p/p\mathbf{Z}_p$. Then we have

$$\bar{M}_A = U \perp \operatorname{Rad} \bar{M}_A$$

with U a non-degenerate quadratic submodule of $\bar{M}_A$. Put $l = l(A) = \operatorname{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} U$. If $l(A)$ is even, put $\bar{\xi}_p(A) = 1$ or -1 according as U is hyperbolic space or not. Here we make the convention that $\bar{\xi}_p(A)=1$ if $l(A)=0$. Then we define Andrianov's polynomial $B_p(v;A)$ as follows:

$$B_p(v,A) = \begin{cases} (1+v)(1-\bar{\xi}_p(A)p^{-l/2}v)\prod_{i=1}^{l/2-1}(1-p^{-2i}v^2) & \text{if } l \text{ is even}, \\ (1+v)\prod_{i=1}^{(l-1)/2}(1-p^{-2i}v^2) & \text{if } l \text{ is odd}. \end{cases}$$

Here we understand that we have $B_p(v, A) = 1$ if $l = 0$. For a non-degenerate half-integral matrix A over $\mathbf{Z}$ put

$$B(s; A) = \prod_p B_p(p^{-s}; A).$$

For a positive definite half-integral matrix A of degree n over $\mathbf{Z}$, put

$$M(A) = \sum_{A' \in \mathcal{G}(A)} \frac{1}{r(A', A')}.$$

Now let A and C be non-degenerate half-integral matrices of degree n over $\mathbf{Z}_p$. We say A dominates C over $\mathbf{Z}_p$ if there is a square matrix D with entries in $\mathbf{Z}_p$ such that $C = A[D]$, and define a polynomial $T_p(u; A, C)$ in u by

$$T_p(u; A, C) = \prod_{i=1}^{m_p} (1 - p^{-n+i}u),$$

where $m_p = 1/2(v_p(\det C) - v_p(\det A))$. We also put $T(s; A, C) = 1$ if A does not dominate C over $\mathbf{Z}_p$. For positive definite half-integral matrices A and C of degree n over $\mathbf{Z}$, we define a finite Euler product $T(s; A, C)$ by

$$T(s; A, C) = \prod_p T_p(p^{-s}; A, C).$$

Now let $\mathbf{H}_n$ be the Siegel's upper half-space. A holomorphic function f on $\mathbf{H}_n$ is called a modular form of weight k belonging to Γ_n, or simply a Siegel modular form of degree n, if it satisfies the following conditions:

(i) $f((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^k f(Z)$ for any $M = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right) \in \Gamma_n$;
(ii) if $n = 1$, for any $\alpha > 0, f(z)$ is bounded on each set $\{x + iy; y \geqslant \alpha\}$.

Assume that f is a Hecke eigenform, namely, that $f(Z)$ is a common eigenfunction of all the Hecke operators (cf. [I-K2]). Then for each prime number p, let $\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p}$ denote the Satake p-parameters of the local Hecke algebra $\mathbf{L}(GSp_n(\mathbf{Q}_p) \cap GL_{2n}(\mathbf{Z}_p), GSp_n(\mathbf{Q}_p))$ determined by f (cf. [I-K2]). We then define the standard zeta function $\zeta^{\mathrm{st}}(f, s)$ of f by

$$\zeta^{\mathrm{st}}(f, s) = \prod_p \left\{ (1 - p^{-s}) \prod_{i=1}^{n} (1 - \alpha_{i,p} p^{-s})(1 - \alpha_{i,p}^{-1} p^{-s}) \right\}^{-1}.$$

Remark. (1) The $Sp_n(\mathbf{Z}_p)$ in Section 3 of [I-K2] should be replaced by $GSp_n(\mathbf{Q}_p) \cap GL_{2n}(\mathbf{Z}_p)$.

(2) In [I-K2], we defined $\zeta^+(f,s)$ by

$$\zeta^+(f,s) = \prod_p \left\{ \prod_{i=1}^n (1-\alpha_{i,p}p^{-s})(1-\alpha_{i,p}^{-1}p^{-s}) \right\}^{-1},$$

which we called the standard zeta function of f. However, from the automorphic representation theoretic view point, it seems more appropriate to call $\zeta^{\mathrm{st}}(f,s)$ the standard zeta function of f.

Now for an element D of $M_n(\mathbf{Z}_p)^\times$, we put $\pi_p(D) = (-1)^i p^{\langle i-1 \rangle}$ or 0 according as D belongs to $GL_n(\mathbf{Z}_p)(E_{n-i} \perp pE_i)GL_n(\mathbf{Z}_p)$ for some $0 \leqslant i \leqslant n$, or not. Here we write $<j> = j(j+1)/2$ for an integer j. Furthermore, for an element D of $M_n(\mathbf{Z})^\times$ put $\pi(D) = \prod_p \pi_p(D)$. This is a certain generalization of the Möbius function (cf. [I-K2]). Let

$$f(Z) = \sum_A c_f(A) \exp(2\pi i \operatorname{tr}(AZ))$$

be the Fourier expansion of $f(Z)$ as in Introduction. Then for an element $A \in \mathscr{H}_n(\mathbf{Z})_{>0}$, following [B-R], we define the A-th *primitive Fourier coefficient* $c_f^*(A)$ of f by

$$c_f^*(A) = \sum_D \pi(D) c_f(A[D^{-1}]),$$

where D runs over a complete set of representatives of left $GL_n(\mathbf{Z})$-equivalence classes of non-degenerate square matrices of degree n with entries in $\mathbf{Z}$. We note that we have

$$c_f(A) = \sum_D c_f^*(A[D^{-1}]),$$

where D runs over a complete set of representatives of left $GL_n(\mathbf{Z})$-equivalence classes of non-degenerate square matrices of degree n with entries in $\mathbf{Z}$ (cf. [I-K2]). Put

$$G_f^*(A) = \sum_{C \in \mathscr{G}(A)} \frac{c_f^*(C)}{r(C,C)}.$$

Furthermore put

$$K(f,s) = \sum_{A_0} \frac{\sigma(A_0)B(2s+1-k, A_0)}{(\det A_0)^s} \times M(A_0) \sum_{C_0} \frac{G(C_0,A_0)G_f^*(C_0)}{M(C_0)} T(2s+2-2k; C_0, A_0),$$

where A_0 and C_0 run over all genera of positive definite half-integral matrices of degree n. Then:

Theorem 2 (Ibukiyama and Katsurada [I-K2, Theorem 3.3]). *Let $L(f,s)$ be the Koecher–Maaß Dirichlet series of f as in Introduction. Then under the above notation and the assumption we have*

$$L(f,s) = \frac{\zeta^{\mathrm{st}}(f,2s-k+1)}{\zeta(2s-k+1)} K(f,s).$$

3. Koecher–Maaß Dirichlet series for Klingen Eisenstein series

Let k,n be positive integers such that $n \leqslant k-3$ and $k \equiv 0 \bmod 2$. Let

$$f(z) = \sum_{e=1}^{\infty} b(e) \exp(2\pi i e z),$$

be a cuspidal Hecke eigenform of weight k belonging to Γ_1, and

$$[f]_1^n(Z) = \sum_{C \in \mathscr{H}_n(\mathbf{Z})_{\geqslant 0}} c_{[f]_1^n}(C) \exp(2\pi i \operatorname{tr}(CZ)).$$

Klingen's Eisenstein series of degree n attached to f. Then rewriting [B-R, Theorem 1] or [Ki1, Theorem] we obtain

Proposition 3.1. *Let $c^*_{[f]_1^n}(C)$ be the C-th primitive Fourier coefficient of $[f]_1^n$ for $C \in \mathscr{H}_n(\mathbf{Z})_{>0}$. Then we have*

$$c^*_{[f]_1^n}(C) = \frac{c^*_{n,k}(C)}{\mu(k)\zeta^{\mathrm{st}}(f,k-1)} \sum_{e=1}^{\infty} \frac{r(C,e)b(e)}{e^{k-1}},$$

where

$$\mu(k) = \frac{(-1)^{k/2} 2^{k+1} \pi^k}{\zeta(k)\zeta(2k-2)\Gamma(k)},$$

and $c^*_{n,k}(C)$ denotes the C-th primitive Fourier coefficient of the Siegel Eisenstein series of degree n and of weight k.

Now for a non-degenerate half-integral matrix A of degree m over $\mathbf{Z}_p$ and a non-degenerate symmetric matrix C of degree n with entries in $\mathbf{Q}_p$, we define the local density $\alpha_p(A,C)$ and the primitive local density $\alpha_p(A,C)^*$ representing C by A as

$$\alpha_p(A,C) = 2^{-\delta_{m,n}} \lim_{e \to \infty} p^{e(-mn+n(n+1)/2)} \# \mathscr{A}_e(A,C),$$

and

$$\alpha_p(A,C)^* = 2^{-\delta_{m,n}} \lim_{e\to\infty} p^{e(-mn+n(n+1)/2)} \#\mathscr{A}_e(A,C)^*,$$

where $\delta_{m,n}$ is Kronecker's delta,

$$\mathscr{A}_e(A,C) = \{X \in M_{mn}(\mathbf{Z}_p)/p^e M_{mn}(\mathbf{Z}_p);\, A[X] - C \in p^e \mathscr{H}_n(\mathbf{Z}_p)\},$$

and

$$\mathscr{A}_e(A,C)^* = \{X \in \mathscr{A}_e(A,C);\, \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} = n\}.$$

Further put

$$G_p(A,C) = \frac{\alpha_p(A,C)}{\alpha_p(A,A)} p^{v_p(\det C)(m-n-1)/2 + v_p(\det A)(m-n+1)/2}.$$

For positive integers k,n put

$$\gamma_{n,k} = (-1)^{nk/2} \pi^{nk+n/2-n^2/2} \varepsilon_n 2^{nk} \frac{\prod_{i=1}^{n-1} \Gamma((n-i)/2)}{\mu(k) \prod_{i=0}^{n-1} \Gamma(k-i/2)}.$$

Then we have

$$\begin{aligned}
K([f]_1^n, s) &= \sum\nolimits_A (\det A)^{-s} M(A) B(2s-k+1, A) \sigma(A) \\
&\quad \times \sum_{C_0} \frac{G(C_0,A) G^*_{[f]_1^n}(C_0)}{M(C_0)} T(2s-2k+2; C_0, A) \\
&= \frac{1}{\mu(k)\zeta^{\mathrm{st}}(f,k-1)} \sum_A (\det A)^{-s} M(A) B(2s-k+1, A)\sigma(A) \\
&\quad \times \sum_{C_0} G(C_0,A) T(2s-2k+2; C_0, A) c^*_{n,k}(C_0) \sum_{e=1}^{\infty} \frac{G(C_0,e)}{M(C_0)} \frac{b(e)}{e^{k-1}} \\
&= \frac{2\gamma_{n,k}}{\zeta^{\mathrm{st}}(f,k-1)} \sum_A (\det A)^{-s+(n+1)/2} \prod_p \alpha_p(A,A)^{-1} \\
&\quad \times \prod_p B_p(p^{-2s+k-1}, A) \prod_p \sigma_p(A) \\
&\quad \times \sum_{C_0} \prod_p G_p(C_0, A) \prod_p T_p(p^{-2s+2k-2}; C_0, A) (\det C_0)^{k-(n+1)/2} \\
&\quad \times \prod_p \alpha_p(H_k, C_0)^* \sum_{e=1}^{\infty} e^{(n-2)/2} (\det C_0)^{-1/2} \prod_p \alpha_p(C_0, e) \frac{b(e)}{e^{k-1}},
\end{aligned}$$

where A and C_0 run over all genera of positive definite half-integral matrices of degree n, and $H_k = \overbrace{\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}}^{k}$. For $A \in \mathscr{H}_n(\mathbf{Z}_p)^\times$ and $e \in \mathbf{Z}_p \backslash \{0\}$ put

$$\begin{aligned} H_p(s; A; e) = & \frac{p^{((n+1)/2-s)\nu(\det A)} \sigma_p(A) B_p(p^{-2s+k-1}; A)}{\alpha_p(A, A)} \\ & \times \sum_{C_0} p^{\nu(\det C_0)(2k-2-n)/2} G_p(C_0, A) \\ & \times\ T_p(p^{-2s+2k-2}; C_0, A) \alpha_p(H_k, C_0)^* \alpha_p(C_0, e), \end{aligned}$$

and for a non-zero p-adic number D_0 and a $GL_n(\mathbf{Z})_p$-invariant function ω_p on $\mathscr{H}_n(\mathbf{Z}_p)^\times$ put

$$H_p(s; D_0; \omega_p, e) = \sum_{l=0}^{\infty} \sum_{A \in \mathscr{A}(D_0, l)} \omega_p(A) H_p(s; A; e),$$

where C_0 runs over a complete set of representatives of $GL_n(\mathbf{Z}_p)$-equivalence classes of half-integral matrices of degree n over $\mathbf{Z}_p$, and $\mathscr{A}(D_0, l)$ denotes a complete set of representatives of $GL_n(\mathbf{Z}_p)$-equivalence classes of half-integral matrices of degree n over $\mathbf{Z}_p$ whose determinant is $p^{2l-2[n/2]\delta_{2,p}} D_0$. We note that $H_p(s; D_0; \omega_p, e)$ is a polynomial in p^{-s}. We also note that the Hasse invariant $h_p(C_0)$ of C_0 is the same as that of A if C_0 dominates A over $\mathbf{Z}_p$. Let A be a positive definite half-integral matrix of degree n over $\mathbf{Z}$. If n is even, then $\det A$ can be expressed as $D_0 m^2$ with positive integers D_0 and m such that $\nu_p(D_0) \leqslant 1$ for $p \neq 2$, and $(-1)^{n/2} D_0 \equiv 1$ or $\equiv 0 \bmod 4$. If n is odd, $\det A$ can be expressed as $D_0 m^2$ with a positive integer m and a square free positive integer D_0. Thus by the same method as in [I-S], we have

Theorem 3.2. (1) *Let n be even. Then we have*

$$\begin{aligned} K([f]_1^n, s) = & \frac{\gamma_{n,k}}{\zeta^{\mathrm{st}}(f; k-1)} \sum_{e=1}^{\infty} \frac{b(e)}{e^{k-n/2}} \\ & \times \sum_{D_0 \in \mathscr{F}_{(-1)^{n/2}}} \left(\prod_p H_p(s; D_0; \iota_p; e) + \prod_p H_p(s; D_0; h_p; e) \right). \end{aligned}$$

(2) *Let n be odd. Then we have*

$$K([f]_1^n, s) = \frac{\gamma_{n,k}}{\zeta^{\mathrm{st}}(f; k-1)} \sum_{e=1}^{\infty} \frac{b(e)}{e^{k-n/2}} \times \sum_{D_0} \left(\prod_p H_p(s; D_0; \iota_p; e) + \prod_p H_p(s; D_0; h_p; e) \right),$$

where D_0 runs over all square free positive integers.

4. Preliminary results

Throughout this section and the next, we assume that n is an even positive integer. In this section we give several lemmas needed for the proof of the main result. Let $p \neq 2$. Let C be an element of $\mathscr{K}_n(\mathbf{Z}_p)$. Then C is equivalent to the following form:

$$U_0 \perp pU_1,$$

where U_0 and U_1 are symmetric unimodular matrices of degree n_0 and n_1, respectively. Here we understand that U_0 or U_1 is the *empty matrix* according as $n_0 = 0$ or $n_1 = 0$. We say that C is type (i,j) if $(n_0, n_1) \equiv (i,j) \bmod 2$. Then B is of type $(0,0)$ or $(1,1)$ since n is even. If C is of type $(0,0)$, we put $\Xi^{(i)}(C) = (-1)^{n_i/2} \det U_i, \xi^{(i)}(C) = \xi_p^{(i)}(C) = \chi_p((-1)^{n_i/2} \det U_i)$ for $i = 0, 1$. Here we understand that we have $\Xi^{(i)}(C) = 1$ and $\xi^{(i)}(C) = 1$ if $n_i = 0$. Then $\Xi^{(i)}(C)$ is uniquely determined, up to $\mathbf{Z}_p^{*\square}$, by C, and thus $\xi^{(i)}(C)$ is uniquely determined by C. Furthermore, if C is of type $(1,1)$, for $i = 0, 1$ and an element e of $\mathbf{Z}_p \backslash \{0\}$ put $\eta^{(i)}(e, C) = \chi_p((-1)^{(n_i-1)/2} e \det U_i)$. The quantity $\eta^{(i)}(e, C)$ is uniquely determined by C and e. Then the following assertion follows from [Y, Theorem 3.1].

Proposition 4.1. *Let $p \neq 2$. Let $C \in \mathscr{K}_n(\mathbf{Z}_p)$, and $e \in \mathbf{Z}_p \backslash 0$. Put $l = [\nu(\det C)/2]$, and $r = \nu(e)$.*

(1) *Let C be of type $(0,0)$, and put $\xi = \xi^{(0)}(C), \delta = \chi_p((-1)^{n/2} \det C)$. Then we have*

$$\alpha_p(C, e) = \delta\xi p^l(\delta\xi p^{-l} - 1) + \delta\xi p^l(1 - \delta p^{-n/2}) \frac{1 - (\delta p^{1-n/2})^{r+1}}{1 - \delta p^{1-n/2}}.$$

(2) *Let C be of type $(1,1)$, and put $\eta = \eta^{(0)}(e, C)$ or $\eta^{(1)}(e, C)$ according as r is even or odd. Then we have*

$$\alpha_p(C, e) = 1 + \eta p^{(r+1)(1-n/2)+l}.$$

Let C be an element of $\mathscr{K}_n(\mathbf{Z}_2)$. Then C is exactly one of the following forms:

$(0,0)$ $\frac{1}{2} U_0 \perp U_1$,
$(1,1)$ $H_{n_0/2} \perp c_0 \perp 2H_{n_1/2} \perp 2c_1$,
$(2,0)$ $H_{n_0/2} \perp V \perp 2H_{n_1/2}$,

where U_0 and U_1 are even-integral unimodular matrices of degree n_0 and n_1, respectively, V is a diagonal unimodular matrix of degree 2 whose determinant is congruent to 1 modulo 4, and c_0, c_1 are 2-adic units. Here we understand that U_0 is the *empty matrix* if $n_0 = 0$, and the others. If C is type (0,0), put $\Xi^{(i)}(C) = (-1)^{n_i/2} \det U_i$ and $\xi^{(i)}(C) = \xi_2^{(i)}(C) = \chi_2((-1)^{n_i/2} \det U_i)$ for $i = 0, 1$. We make the convention that we have $\Xi^{(i)}(C) = 1$ and $\xi^{(i)}(C) = 1$ if $n_i = 0$. The quantity $\Xi^{(i)}(C)$ is uniquely determined, up to $\mathbf{Z}_2^{*\square}$, by C, and $\xi^{(i)}(C)$ is uniquely determined by C. Furthermore, for an element e of $\mathbf{Z}_p \backslash \{0\}$, put $\eta(e, C) = h(-e \perp c_0 \perp 2c_1)$ or $h(V \perp -e)$ according as the case (1,1) or (2,0), where h is the Hasse invariant on $\mathbf{Q}_2$. The quantity $\eta(e, C)$ is uniquely determined by C and e. Then similarly to Proposition 4.1, the following assertion follows from [Y, Theorem 4.1].

Proposition 4.2. *Let $C \in \mathscr{H}_n(\mathbf{Z}_2)$, and $e \in \mathbf{Z}_2 \backslash 0$. Put $l = [v(2^n \det C)/2]$, and $r = v(e)$.*

(1) *Let C be of type $(0,0)$, and put $\xi = \xi^{(0)}(C), \delta = \chi_2((-1)^{n/2} \det C)$. Then we have*

$$\alpha_2(C, e) = \delta\xi 2^l(\delta\xi 2^{-l} - 1) + \delta\xi 2^l(1 - \delta 2^{-n/2}) \frac{1 - (\delta 2^{1-n/2})^{r+1}}{1 - \delta 2^{1-n/2}}.$$

(2) *Let C be of type $(1,1)$ or $(2,0)$, and put $\eta = \eta(e, C)$. Then we have*

$$\alpha_2(C, e) = 1 + \eta 2^{(r+l_0)(1-n/2)+l},$$

where l_0 is 3 or 2 according as C is type $(1,1)$ or $(2,0)$.

For a non-negative integer m we define a polynomial $\varphi_m(x)$ in x by $\varphi_m(x) = \prod_{i=1}^m (1 - x^i)$. The next proposition is merely a reformulation of [I-K2, Theorem 2.10].

Proposition 4.3. *Let A and C_0 be non-degenerate half-integral matrices of degree n over $\mathbf{Z}_p$. Assume that A belongs to $\mathscr{H}_n(\mathbf{Z}_p)$ and C_0 dominates A. Put $l = [v(2^n \det C_0)/2]$ and $m = \frac{v(\det A) - v(\det C_0)}{2}$.*

(1) *Let $p \neq 2$. Then we have*

$$G_p(C_0, A) = \begin{cases} \dfrac{p^{2lm+m(m-1)/2}(1 + \xi^{(1)}(A)p^{-l}) \prod_{i=1}^m (1 - p^{-2l-2m+2i-2})}{(1 + \xi^{(1)}(A)p^{-l-m})\varphi_m(p^{-1})} & \text{if } C_0 \text{ is type } (0,0), \\[2ex] \dfrac{p^{2lm+m(m+1)/2} \prod_{i=1}^m (1 - p^{-2l-2m+2i-2})}{\varphi_m(p^{-1})} & \text{if } C_0 \text{ is type } (1,1). \end{cases}$$

(2) *Let $p = 2$. Then we have*

$$G_2(C_0, A)$$
$$= \begin{cases} \dfrac{2^{2lm+m(m-1)/2}(1+\xi^{(1)}(A)2^{-l})\prod_{i=1}^{m}(1-2^{-2l-2m+2i-2})}{(1+\xi^{(1)}(A)2^{-l-m})\varphi_m(2^{-1})} & \text{if } C_0 \text{ is type } (0,0), \\ \dfrac{2^{2(l-1)m+m(m+1)/2}\prod_{i=1}^{m}(1-2^{-2l-2m+2i})}{\varphi_m(2^{-1})} & \text{if } C_0 \text{ is type } (1,1), \\ \dfrac{2^{2(l-1)m+m(m+1)/2}\prod_{i=1}^{m}(1-2^{-2l-2m+2i})}{\varphi_m(2^{-1})} & \text{if } C_0 \text{ is type } (2,0). \end{cases}$$

The following is well known (cf. [Ki3, Theorem 5.6.3]).

Proposition 4.4. *Let A be an element of $\mathscr{K}_n(\mathbf{Z}_p)$. Let $l = [v(2^n \det A)/2]$.*
(1) *Let $p \neq 2$. Then we have*

$$\alpha_p(A, A) = \begin{cases} \dfrac{2p^{l(2l+1)}\varphi_l(p^{-2})\varphi_{n/2-l}(p^{-2})}{(1+\xi^{(1)}(A)p^{-l})(1+\xi^{(0)}(A)p^{-n/2+l})} & \text{if } A \text{ is type } (0,0), \\ 2p^{(l+1)(2l+1)}\varphi_l(p^{-2})\varphi_{n/2-l-1}(p^{-2}) & \text{if } A \text{ is type } (1,1). \end{cases}$$

(2) *Let $p = 2$. Then we have*

$$\alpha_2(A, A) = \begin{cases} \dfrac{2^{l(2l+1)+1}\varphi_l(2^{-2})\varphi_{n/2-l}(2^{-2})}{(1+\xi^{(1)}(A)2^{-l})(1+\xi^{(0)}(A)2^{-n/2+l})} & \text{if } A \text{ is type } (0,0), \\ 2^{l(2l-1)+3}\varphi_{l-1}(2^{-2})\varphi_{n/2-l}(2^{-2}) & \text{if } A \text{ is type } (1,1), \\ 2^{l(2l-1)+2}\varphi_{l-1}(2^{-2})\varphi_{n/2-l}(2^{-2}) & \text{if } A \text{ is type } (2,0). \end{cases}$$

The following proposition follows from [Ki2, Lemma 9].

Proposition 4.5. *Let C be an element of $\mathscr{K}_n(\mathbf{Z}_p)$. Put $l = [v(2^n \det C)/2]$.*
(1) *Let $p \neq 2$. Then we have*

$$\alpha_p(H_k, C)^* = (1-p^{-k})$$
$$\times \begin{cases} \prod_{i=1}^{n/2+l-1}(1-p^{-2k+2i})(1+\xi^{(0)}(C)p^{-k+n/2+l}) & \text{if } C \text{ is type } (0,0), \\ \prod_{i=1}^{n/2+l}(1-p^{-2k+2i}) & \text{if } C \text{ is type } (1,1). \end{cases}$$

(2) *Let* $p = 2$. *Then we have*

$$\alpha_2(H_k, C)^* = (1 - 2^{-k}) \times \begin{cases} \prod_{i=1}^{n/2+l-1}(1 - 2^{-2k+2i})(1 + \xi^{(0)}(C)2^{-k+n/2+l}) & \text{if } C \text{ is type } (0,0), \\ \prod_{i=1}^{n/2+l}(1 - 2^{-2k+2i}) & \text{if } C \text{ is type } (1,1), \\ \prod_{i=1}^{n/2+l-1}(1 - 2^{-2k+2i}) & \text{if } C \text{ is type } (2,0). \end{cases}$$

Before proceeding the proof of our main result, we need the following combinatorial lemma.

Let X, Y, w be variables, and N be a positive integer. Then for integers l, m put

$$Q(X, Y, w; N, l, m) = \prod_{i=1}^{l} (1 - w^{2i-2}X^{-1}Y) \prod_{i=1}^{N-l-m} (1 - w^{2-2i-2N}XY) \times \prod_{i=1}^{m}(1 - w^{-2N+i}X)\frac{\varphi_N(w^{-2})(-1)^{l+m}w^{-l^2+l}w^{-m^2/2+m/2}X^l Y^m}{\varphi_l(w^{-2})\varphi_{N-l-m}(w^{-2})\varphi_m(w^{-1})}.$$

For integers i and j, put

$$Q_{ij}(X, Y, w; N, l, m) = Q(X, Y, w; N, l, m)w^{li+mj},$$

and

$$Q_{ij}(X, Y, w; N) = \sum_{l=0}^{N} \sum_{m=0}^{N-l} Q_{ij}(X, Y, w; N, l, m).$$

Further put

$$Q(X, Y, w; N) = \prod_{i=0}^{N-1}(1 - w^{2i-2N+2}X)(1 - w^{2i-2N+1}Y).$$

Lemma 4.6. (1) *For any* i, j *we have*

(1.1) $$Q_{i,j-1}(X, Y, w; N) = Q_{ij}(X, Y, w; N) + (1 - w^{-2N}) \times (1 - w^{-2N+1}X)w^j YQ_{i+1,j}(w^{-1}X, w^{-1}Y, w; N - 1),$$

(1.2) $$Q_{i-2,j}(X, Y, w; N) = Q_{ij}(X, Y, w; N) + (1 - w^{-2N}) \times (1 - X^{-1}Y)w^i XQ_{i,j}(w^{-2}X, Y, w; N - 1).$$

(2) We have

(2.1) $Q_{0,0}(X, Y, w; N) = Q(X, Y, w; N)$,

(2.2) $Q_{-1,0}(X, Y, w; N) = Q(w^{-1}X, wY, w; N)$,

(2.3) $Q_{-2,-1}(X, Y, w; N) = Q(w^{-2}X, Y, w; N)$,
(2.4) $Q_{-1,-1}(X, Y, w; N) = Q(w^{-1}X, w^{-1}Y, w; N)$.

Proof. Assertion (1) can easily be shown by a direct calculation. We prove assertion (2) by induction on N. It is easy to show that assertions (2.1)–(2.4) are true for $N = 1$. Let $N > 1$ and assume that the assertions hold for any positive integer $N' < N$. For integers K, l, m put

$$G(K, l, m) = \frac{\varphi_K(w^{-2})}{\varphi_l(w^{-2})\varphi_{K-l-m}(w^{-2})\varphi_m(w^{-1})},$$

and

$$Q'(X, Y, w; K, l, m) = \prod_{i=1}^{l} (1 - w^{2i-2}X^{-1}Y) \prod_{i=1}^{K-l-m} (1 - w^{2-2i-2K}XY) \times \prod_{i=1}^{m} (1 - w^{-2K+i}X)(-1)^{l+m} w^{-l^2+l} w^{-m^2/2+m/2} X^l Y^m.$$

Further for integers i,j put

$$R_{i,j}(X, Y, w; N) = \sum_{l=0}^{N} \sum_{m=0}^{N-l} Q'(X, Y; N, l, m) w^{li+mj} G(N-1, l-1, m),$$

$$S_{i,j}(X, Y, w; N) = \sum_{l=0}^{N} \sum_{m=0}^{N-l} Q'(X, Y; N, l, m) w^{l(i-2)+mj} G(N-1, l, m-1),$$

$$T_{i,j}(X, Y, w; N) = \sum_{l=0}^{N} \sum_{m=0}^{N-l} Q'(X, Y; N, l, m) w^{l(i-2)+m(j-1)} G(N-1, l, m-1),$$

and

$$U_{i,j}(X, Y, w; N) = \sum_{l=0}^{N} \sum_{m=0}^{N-l} Q'(X, Y; N, l, m) w^{l(i-2)+m(j-2)} G(N-1, l, m).$$

Here we understand $G(N-1, l-1, m) = 0$ if $l = 0$ or $m = N$, and the others.

Then we have

$$G(N, l, m) = G(N-1, l-1, m) + w^{-2l} G(N-1, l, m-1) + w^{-2l-m} G(N-1, l, m-1) + w^{-2l-2m} G(N-1, l, m).$$

Thus we have

$$Q_{i,j}(X,Y,w;N) = R_{i,j}(X,Y;N) + S_{i,j}(X,Y;N) + T_{i,j}(X,Y;N) + U_{i,j}(X,Y;N).$$

On the other hand, it can easily be seen that we have

$$R_{i,j}(X,Y,w;N) = -(1-X^{-1}Y)w^i XQ_{i,j}(w^{-2}X,Y,w;N-1),$$

$$S_{i,j}(X,Y,w;N) = -(1-w^{-2N+1}X)w^j YQ_{i-1,j}(w^{-1}X,w^{-1}Y,w;N-1),$$

$$T_{i,j}(X,Y,w;N) = -(1-w^{-2N+1}X)w^{j-1} YQ_{i-1,j-1}(w^{-1}X,w^{-1}Y,w;N-1),$$

and

$$U_{i,j}(X,Y,w;N) = (1-w^{-2N}XY)Q_{i,j}(w^{-2}X,w^{-2}Y,w;N-1).$$

Thus we have

$$\begin{aligned} Q_{0,0}(X,Y,w;N) = & -(1-X^{-1}Y)XQ_{0,0}(w^{-2}X,Y,w,N-1) \\ & -(1-w^{-2N+1}X)YQ_{-1,0}(w^{-1}X,w^{-1}Y,w;N-1) \\ & -(1-w^{-2N+1}X)w^{-1}YQ_{-1,-1}(w^{-1}X,w^{-1}Y,w;N-1) \\ & +(1-w^{-2N}XY)Q_{0,0}(w^{-2}X,w^{-2}Y,w;N-1). \end{aligned}$$

Thus by the induction hypothesis we have

$$Q_{0,0}(X,Y,w;N) = \prod_{i=0}^{N-1}(1-w^{2i-2N+2}X)(1-w^{2i-2N+1}Y).$$

Thus assertion (2.1) holds for N. Similarly assertion (2.4) holds for N. Further by (1.1) we have

$$\begin{aligned} Q_{-1,0}(X,Y,w;N) = & \, Q_{-1,-1}(X,Y,w;N) \\ & -(1-w^{-2N})Y(1-w^{-2N+1}X)Q_{0,0}(w^{-1}X,w^{-1}Y,w;N-1). \end{aligned}$$

Thus by the induction hypothesis and the fact proved just before, we have

$$Q_{-1,0}(X,Y,w;N) = \prod_{i=0}^{N-1}(1-w^{2i-2N+1}X)(1-w^{2i-2N+2}Y).$$

Thus assertion (2.2) holds for N. Finally by (1.1) and (1.2) we have

$$\begin{aligned}
Q_{-2,-1}(X,Y,w;N) &= Q_{-2,0}(X,Y,w;N) + (1-w^{-2N}) \\
&\quad \times Y(1-w^{-2N+1}X)Q_{-1,0}(w^{-1}X, w^{-1}Y, w; N-1) \\
&= Q_{0,0}(X,Y,w;N) + (1-w^{-2N}) \\
&\quad \times X(1-X^{-1}Y)Q_{0,0}(w^{-2}X, Y, w; N-1) \\
&\quad + (1-w^{-2N})Y(1-w^{-2N+1}X)Q_{0,0}(w^{-2}X, Y, w; N-1).
\end{aligned}$$

Thus by the induction hypothesis and the fact proved just before, we have

$$Q_{-2,-1}(X,Y,w;N) = \prod_{i=0}^{N-1}(1-w^{2i-2N}X)(1-w^{2i-2N+1}Y).$$

This completes the induction. $\square$

The following lemma will be used for the proof of Theorem 5.2.

Lemma 4.7. *Under the same notation as above, we have*

$$\begin{aligned}
(1)\ Q_{-1,0}(X^2,Y^2,w;N) &= w^{2N}Q_{-1,-1}(X^2,Y^2,w;N) - w^{2N}(1-w^{-2N}) \\
&\quad \times (1-w^{-2N+1}X^2)Q_{0,0}(w^{-1}X^2, w^{-1}Y^2, w; N-1).
\end{aligned}$$

$$\begin{aligned}
(2)\ Q_{i,0}(X^2,Y^2,w;N) &= w^{2N}Q_{i-2,-1}(X^2,Y^2,w;N) - w^{2N}(1-w^{-2N}) \\
&\quad \times (1-w^{-2N+1}X^2)Q_{i-1,0}(w^{-1}X^2, w^{-1}Y^2, w; N-1) \\
&\quad - wX^2(1-w^{-2N})(1-X^{-2}Y^2)Q_{i,0}(w^{-2}X^2, Y^2, w; N-1)
\end{aligned}$$

for $i = 0, -1$.

Proof. The both identities can be proved by Lemma 4.6. $\square$

5. Proof of the main result

We denote by $\mathscr{K}_n(\mathbf{Z}_p)^{(i,j)}$ the subset of $\mathscr{K}_n(\mathbf{Z}_p)$ consisting of all matrices of type (i,j). Furthermore for two elements $a, b \in \mathbf{Q}_p$, let $(a,b)_p$ denote the Hilbert symbol on $\mathbf{Q}_p$. The following lemma is well-known (cf. [I-K2, Lemma 2.2]).

Lemma 5.1. *Let* $C \in \mathcal{H}_n(\mathbf{Z}_p)$.

(1) *Let* $p \neq 2$, *and* $C \sim U_0 \perp pU_1$ *with* U_0 *and* U_1 *unimodular matrices of degree* n_0 *and* n_1, *respectively. Then we have*

$$h_p(C) = \begin{cases} \chi_p((-1)^{n_1/2} \det U_1) & \text{if } n_0 \text{ even}, \\ \chi_p((-1)^{(n_1+1)/2} \det U_0) & \text{if } n_0 \text{ odd}. \end{cases}$$

(2) *Let* $p = 2$.

(2.1) *Let* $C \sim 1/2U_0 \perp U_1$ *with* U_0 *and* U_1 *even unimodular matrices of degree* n_0 *and* n_1, *respectively. Then we have*

$$h_2(C) = (-1)^{n(n+2)/8} \chi_2((-1)^{n_1/2} \det U_1).$$

(2.2) *Let* $C \sim 1/2U_0 \perp U_1 \perp W$, *where* U_0 *and* U_1 *are even unimodular matrices of degree* n_0 *and* n_1, *respectively, and* $W = c_0 \perp 2c_1$ with $c_0, c_1 \in \mathbf{Z}_2^*$ *or* W *is a diagonal unimodular matrix of degree* 2 *such that* $\det W \equiv 1 \bmod 4$. *Then we have*

$$h_2(C) = (-1)^{n(n-2)/8}((-1)^{n/2-1}, \det W)_2 h_2(W).$$

From now on for two elements a and b of $\mathbf{Z}_p^*$ we write $a \sim b$ if $ab^{-1} \in \mathbf{Z}_p^{*\square}$. Further we use the same symbol a to denote the equivalence class of a in $\mathbf{Z}_p^*/\mathbf{Z}_p^{*\square}$. We denote by $\mathcal{D} = \mathcal{D}_p$ the set $\{1, 5\}$ or the complete set of representatives of $\mathbf{Z}_p^*/\mathbf{Z}_p^{*\square}$ according as $p = 2$ or not.

Theorem 5.2. *Let* $D_0 \in \mathbf{Z}_p^*$ *with p odd, or* $D_0 \in \mathbf{Z}_2^*$ *such that* $(-1)^{n/2} D_0 \equiv 1 \bmod 4$. *For a positive integer e put*

$$R_p(e, D_0) = \frac{1 - (\delta p^{1-n/2})^{\nu(e)+1}}{1 - \delta p^{1-n/2}},$$

where $\delta = \delta_p = \chi_p((-1)^{n/2} D_0)$. *Further put*

$$\Upsilon_{n,k} = \Upsilon_{n,k}(p) = \frac{(1 - p^{-k}) \prod_{i=1}^{n/2-1}(1 - p^{-2k+2i})}{\varphi_{n/2-1}(p^{-2})}.$$

Then we have

$$(1)\ H_p(s; D_0; \iota_p, e) = 2^{\delta_{2,p} n(s-k+1/2)} \Upsilon_{n,k}(1 + \delta p^{n/2-k})$$
$$\times \Big[p^{-2s+2k-3}(1 - \delta p^{n/2-k})(1 + p^{-k+2})$$

$$\times \prod_{i=0}^{n/2-2} (1-p^{2i-n-1+2k-2s})(1-p^{2i+2-2s})$$

$$\left. + R_p(e, D_0) \prod_{i=0}^{n/2-1} (1-p^{2i-n-1+2k-2s})(1-p^{2i-2s}) \right].$$

(2) $$H_p(s; D_0; h_p, e) = (-1,-1)_p^{n(n+2)/8} 2^{\delta_{2,p} n(s-k+1/2)} \Upsilon_{n,k} (1+\delta p^{n/2-k})$$

$$\times \left[\delta p^{-2s+2k-n/2-2}(1-\delta p^{n/2-k})(1+p^{n-k}) \right.$$

$$\times \prod_{i=0}^{n/2-2} (1-p^{2i-n+2k-2s})(1-p^{2i+1-2s})$$

$$+ R_p(e, D_0) \left\{ \prod_{i=0}^{n/2-1} (1-p^{2i-n+2k-2s})(1-p^{2i-1-2s}) \right.$$

$$+ (1+p^{-k+1}) p^{-2s+2k-2} (1-\delta p^{n/2-k})(1-\delta p^{-n/2})$$

$$\left. \left. \times \prod_{i=0}^{n/2-2} (1-p^{2i-n+2k-2s})(1-p^{2i+1-2s}) \right\} \right].$$

Proof. Put $D_0 = 2^n d_0$ or d_0 according as $p = 2$ or not. Now let A be an element of $\mathscr{K}_n(\mathbf{Z}_p)$ and put $\Xi = \Xi^{(0)}(A)$. Then we have $G_p(C_0, A) \neq 0$ if and only if $C_0 \in \mathscr{K}_n(\mathbf{Z}_p)$, $(\det A)(\det C_0)^{-1} \in \mathbf{Z}_p^{\square}$, and $\Xi^{(0)}(C_0) \sim \Xi$. Further the $GL_n(\mathbf{Z}_p)$-equivalence class of $A \in \mathscr{K}_n(\mathbf{Z}_p)$ is uniquely determined by its determinant d and Ξ. Thus we denote by $A(d; \Xi)$ its representative. We note that the p-adic order of the determinant of $A \in \mathscr{K}_n(\mathbf{Z}_p)$ is at most $n/2$. Thus for $\omega_p = \iota_p$ or h_p we have

$$H_p(s; D_0; \omega_p, e)$$

$$= 2^{\delta_{2,p} n(s-k+1/2)}$$

$$\times \left[\alpha_p(H_k, A(d_0; \Delta))^* \alpha_p(A(d_0, \Delta), e) \right.$$

$$\times T_p(p^{-2s+2k-2}; A(d_0; \Delta), A(d_0; \Delta)) G_p(A(d_0; \Delta), A(d_0; \Delta))$$

$$\times \frac{\omega_p(A(d_0; \Delta) \sigma_p(A(d_0; \Delta)) B_p(p^{-2s+k-1}; A(d_0; \Delta))}{\alpha_p(A(d_0; \Delta), A(d_0; \Delta))}$$

$$+ \sum_{\Xi \in \mathscr{D}} \sum_{1 \leqslant l+m \leqslant n/2-1} p^{(2k-2-n)l} \alpha_p(H_k, A(p^{2l} d_0; \Xi))^* \alpha_p(A(p^{2l} d_0; \Xi), e)$$

$$\times T_p(p^{-2s+2k-2}; A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))G_p(A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))$$

$$\times \frac{p^{(n+1)(l+m)}p^{-(2l+2m)s}\omega_p(A(p^{2l+2m}d_0;\Xi))\sigma_p(A(p^{2l+2m}d_0;\Xi))B_p(p^{-2s+k-1}; A(p^{2l+2m}d_0;\Xi))}{\alpha_p(A(p^{2l+2m}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))}$$

$$+ \sum_{l=0}^{n/2} p^{(2k-2-n)l}\alpha_p(H_k, A(p^{2l}d_0;1))^*\alpha_p(A(p^{2l}d_0,1),e)$$

$$\times T_p(p^{-2s+2k-2}; A(p^{2l}d_0;1), A(p^n d_0;1))G_p(A(p^{2l}d_0;1), A(p^n d_0;1))$$

$$\left.\times \frac{p^{(n+1)n/2}p^{-ns}\omega_p(A(p^n d_0;1))\sigma_p(A(p^n d_0;1))B_p(p^{-2s+k-1}; A(p^n d_0;1))}{\alpha_p(A(p^n d_0;1), A(p^n d_0;1))}\right],$$

where $\Delta = (-1)^{n/2}d_0$. For integers l, m let

$$\begin{aligned} Q(X,Y,w;n/2,l,m) = {} & \prod_{i=1}^{l}(1-w^{2i-2}X^{-1}Y) \\ & \times \prod_{i=1}^{n/2-l-m}(1-w^{2-2i-n}XY)\prod_{i=1}^{m}(1-w^{-n+i}X) \\ & \times \frac{\varphi_{n/2}(w^{-2})(-1)^{l+m}w^{-l^2+l}w^{-m^2/2+m/2}X^lY^m}{\varphi_l(w^{-2})\varphi_{n/2-l-m}(w^{-2})\varphi_m(w^{-1})} \end{aligned}$$

be the rational function in X, Y and w in Lemma 4.6, and put

$$\begin{aligned} \tilde{Q}(X,Y,w;n/2,l,m) = {} & \prod_{i=1}^{l}(1-w^{2i-2}X^{-1}Y) \\ & \times \prod_{i=1}^{n/2-l-m-1}(1-w^{2-2i-n}XY)\prod_{i=1}^{m}(1-w^{-n+i}X) \\ & \times \frac{\varphi_{n/2}(w^{-2})(-1)^{l+m}w^{-l^2+l}w^{-m^2/2+m/2}X^lY^m}{\varphi_l(w^{-2})\varphi_{n/2-l-m}(w^{-2})\varphi_m(w^{-1})}. \end{aligned}$$

Further put $u = p^{-s+k-1}$ and $v = p^{-s+n/2-1}$, and

$$\Phi_{n,k} = \frac{(1-p^{-k})\prod_{i=1}^{n/2-1}(1-p^{-2k+2i})}{\varphi_{n/2}(p^{-2})}.$$

Then by Propositions 4.3, 4.4, and 4.5, we have

$$\begin{aligned}
&p^{(2k-2-n)l}\alpha_p(H_k, A(p^{2l}d_0;\Xi))^*\\
&\quad\times T_p(p^{-2s+2k-2}; A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))G_p(A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))\\
&\quad\times \frac{p^{(n+1)(l+m)}p^{-(2l+2m)s}\sigma_p(A(p^{2l+2m}d_0;\Xi))B_p(p^{-2s+k-1}; A(p^{2l+2m}d_0;\Xi))}{\alpha_p(A(p^{2l+2m}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))}\\
&\quad= \frac{\Phi_{n,k}}{2}p^{-2l}\tilde{Q}(u^2, v^2, p; n/2, l, m)\delta\xi(1+\delta\xi p^{-l})(1+\xi p^{-n/2+l+m})(1+\xi p^l u^{-1}v)\\
&\qquad\times(1-\xi p^{-n+m+l+1}uv)(1+p^{-n/2+1}uv)
\end{aligned}$$

or

$$= \Phi_{n,k}p^{-2l}Q(u^2, v^2, p; n/2, l, n/2-l)\delta(1+\delta p^{-l})(1+p^l u^{-1}v)$$

according as $l+m \leqslant n/2-1$ or $l+m = n/2$, where $\xi = \chi_p(\Xi)$. Here we understand $\xi = \delta$ or 1 if $l = m = 0$ or if $l+m = n/2$. Further by Propositions 4.1 and 4.2 we have

$$\begin{aligned}
&\delta\xi p^{-2l}(1+\delta\xi p^{-l})\alpha_p(A(p^{2l}d_0;\Xi), e)\\
&\quad= (p^{-2l}-1)p^{-l} + (1-\delta p^{-n/2})R_p(e, D_0)(1+\delta\xi p^{-l})p^{-l}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
H_p(s; D_0; \omega_p, e) = 2^{\delta_{2,p}n(s-k+1/2)}\Phi_{n,k}\big[&\tilde{Q}(u^2, v^2, p; n/2, 0, 0)(1-p^{-n})(1+\delta u^{-1}v)\\
&\times(1-\delta p^{-n+1}uv)(1+p^{-n/2+1}uv)R_p(e, D_0)\omega_p(A(p^{2l+2m}d_0, \Delta))\\
&+ 1/2\sum_{\Xi\in\mathscr{D}}\ \sum_{1\leqslant l+m\leqslant n/2-1} p^{-l}\tilde{Q}(u^2, v^2, p; n/2, l, m)(1+\xi p^{-n/2+l+m})\\
&\times(1+\xi p^l u^{-1}v)(1-\xi p^{-n+m+l+1}uv)(1+p^{-n/2+1}uv)\\
&\times\{(p^{-2l}-1)+(1-\delta p^{-n/2})(1+\delta\xi p^{-l})R_p(e, D_0)\}\\
&\times\omega_p(A(p^{2l+2m}d_0, \Xi)) + \sum_{l=0}^{n/2} p^{-l}Q(u^2, v^2, p; n/2, l, n/2-l)\\
&\times(1+p^l u^{-1}v)\{(p^{-2l}-1)+(1-\delta p^{-n/2})\\
&\times(1+\delta p^{-l})R_p(e, D_0)\}\omega_p(A(p^{2l+2m}d_0, 1))\big].
\end{aligned}$$

For any $0 \leqslant l+m \leqslant n/2-1$ we have

$$
\begin{aligned}
&(1+\xi p^{-n/2+l+m})(1+\xi p^{l}u^{-1}v)(1-\xi p^{-n+m+l+1}uv)(1+p^{-n/2+1}uv)\\
&\quad = 1-p^{-2n+2l+2m+2}u^2v^2 + p^{n/2-m}u^{-1}v(1-p^{-2n+2l+2m+2}u^2v^2)\\
&\qquad - p^{n/2-m}u^{-1}v(1-p^{-n+m+1}u^2)(1-p^{-n+2l+2m})\\
&\qquad + \xi[u^{-1}vp^{l}(1-p^{-2n+2l+2m+2}u^2v^2) + p^{n/2-l-m}(1-p^{-2n+2l+2m+2}u^2v^2)\\
&\qquad - p^{n/2-l-m}\{(1-p^{-n+m+1}u^2)+p^{-n+m+1}u^2(1-p^{2l}u^{-2}v^2)\}(1-p^{-n+2l+2m})].
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\tilde{Q}(u^2,v^2,p;n/2,l,m)(1-p^{-2n+2l+2m+2}u^2v^2) = Q(u^2,v^2,p;n/2,l,m),\\
&\tilde{Q}(u^2,v^2,p;n/2,l,m)(1-p^{-n+m+1}u^2)(1-p^{-n+2l+2m})\\
&\quad = (1-p^{-n})(1-p^{-n+1}u^2)p^{l}p^{m}Q(p^{-1}u^2,p^{-1}v^2,p;n/2-1,l,m),
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{Q}(u^2,v^2,p;n/2,l,m)(1-p^{2l}u^{-2}v^2)(1-p^{-n+2l+2m})\\
&\quad = (1-p^{-n})(1-u^{-2}v^2)p^{2l}Q(p^{-2}u^2,v^2,p;n/2-1,l,m).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
H_p(s;D_0;\iota,e) = 2^{\delta_{2,p}n(s-k+1/2)}\Phi_{n,k}[&\{(Q_{-3,0}(u^2,v^2,p;n/2)-Q_{-1,0}(u^2,v^2,p;n/2))\\
&+p^{n/2}u^{-1}v(Q_{-3,-1}(u^2,v^2,p;n/2)-Q_{-1,-1}(u^2,v^2,p;n/2))\\
&-p^{n/2}u^{-1}v(1-p^{-n})(1-p^{-n+1}u^2)(Q_{-2,0}(p^{-1}u^2,p^{-1}v^2,p;n/2-1)\\
&-Q_{0,0}(p^{-1}u^2,p^{-1}v^2,p;n/2-1))\}\\
&+(1-\delta p^{-n/2})R_p(e,D_0)\{Q_{-1,0}(u^2,v^2,p;n/2)\\
&+p^{n/2}u^{-1}vQ_{-1,-1}(u^2,v^2,p;n/2)\\
&-p^{n/2}u^{-1}v(1-p^{-n})(1-p^{-n+1}u^2)Q_{0,0}(p^{-1}u^2,p^{-1}v^2,p;n/2-1)\}\\
&+\delta(1-\delta p^{-n/2})R_p(e,D_0)\{u^{-1}vQ_{-1,0}(u^2,v^2,p;n/2)\\
&+p^{n/2}Q_{-3,-1}(u^2,v^2,p;n/2)-p^{n/2}(1-p^{-n})
\end{aligned}
$$

$$\times (1 - p^{-n+1}u^2)Q_{-2,0}(p^{-1}u^2, p^{-1}v^2, p; n/2 - 1)$$

$$- p^{-n/2+1}u^2(1 - p^{-n})(1 - u^{-2}v^2)Q_{-1,0}(p^{-2}u^2, v^2, p; n/2 - 1)\}].$$

Thus assertion (1) follows from (1.2) and (2.2) of Lemmas 4.6 and 4.7.

Next let $\omega_p = h_p$. Note that we have $h_p(A(p^{2l}d_0, \Xi)) = (-1,-1)_p^{n(n+2)/8}\delta\xi$ for any $l \geqslant 0$. Thus similarly to the case of $\omega_p = \iota_p$, we have

$$\begin{aligned}
H_p(s; D_0; h_p, e) = {} & (-1,-1)_p^{n(n+2)/8} 2^{\delta_{2,p} n(s-k+1/2)} \Phi_{n,k} \\
& \times [(1 - \delta p^{-n/2})R_p(e, D_0)\{Q_{-2,0}(u^2, v^2, p; n/2) \\
& + p^{n/2}u^{-1}vQ_{-2,-1}(u^2, v^2, p; n/2) \\
& - p^{n/2}(1 - p^{-n})(1 - p^{-n+1}u^2)u^{-1}vQ_{-1,0}(p^{-1}u^2, p^{-1}v^2, p; n/2 - 1)\} \\
& + \delta(1 - \delta p^{-n/2})R_p(e, D_0)\{u^{-1}vQ_{0,0}(u^2, v^2, p; n/2) \\
& + p^{n/2}Q_{-2,-1}(u^2, v^2, p; n/2) - p^{n/2}(1 - p^{-n}) \\
& \times (1 - p^{-n+1}u^2)Q_{-1,0}(p^{-1}u^2, p^{-1}v^2, p; n/2 - 1) \\
& - p^{-n/2+1}u^2(1 - p^{-n})(1 - u^{-2}v^2)Q_{0,0}(p^{-2}u^2, v^2, p; n/2 - 1)\} \\
& + \delta\{u^{-1}v(Q_{-2,0}(u^2, v^2, p; n/2) - Q_{0,0}(u^2, v^2, p; n/2)) \\
& + p^{n/2}(Q_{-4,-1}(u^2, v^2, p; n/2) - Q_{-2,-1}(u^2, v^2, p; n/2)) \\
& - p^{n/2}(1 - p^{-n})(1 - p^{-n+1}u^2)(Q_{-3,0}(p^{-1}u^2, p^{-1}v^2, p; n/2 - 1) \\
& - Q_{-1,0}(p^{-1}u^2, p^{-1}v^2, p; n/2 - 1)) \\
& - p^{-n/2+1}u^2(1 - p^{-n})(1 - u^{-2}v^2)(Q_{-2,0}(p^{-2}u^2, v^2, p; n/2 - 1) \\
& - Q_{0,0}(p^{-2}u^2, v^2, p; n/2 - 1))\}].
\end{aligned}$$

Thus assertion (2) can also be proved by (1.2) and (2.1) of Lemmas 4.6 and 4.7. $\square$

Theorem 5.3. *Let $D_0 \in p\mathbf{Z}_p^*$ with p odd, or $D_0 \in 4\mathbf{Z}_2^*$ such that $(-1)^{n/2}4^{-1}D_0 \equiv 3 \bmod 4$ or $D_0 \in 8\mathbf{Z}_2^*$. Put $l_0 = \nu(D_0)$. Let $e = p^r e_0$ be a positive integer with $(p, e_0) = 1$.*

(1) *We have*

$$H_p(s;D_0;\iota_p,e)=2^{\delta_{2,p}n(s-k+1/2)}p^{(-s+k-3/2)l_0}\Upsilon_{n,k}(1-p^{n-2k})$$
$$\times(1+p^{-2s+k-1})\prod_{i=0}^{n/2-2}(1-p^{2i-n-1+2k-2s})(1-p^{2i+2-2s}).$$

(2) *Put*

$$R_p(e,D_0)=(-1,-1)_p^{n(n-2)/8}(-e,-e)_p((-1)^{n/2}e,-(-1)^{n/2}D_0)_p p^{(1-n/2)r}.$$

Then we have

$$H_p(s;D_0;h_p,e)=\Upsilon_{n,k}2^{\delta_{2,p}n(s-k+1/2)}R_p(e,D_0)p^{(-s+k-(n+1)/2)l_0}(1-p^{n-2k})$$
$$\times(1+p^{-2s+k-1})\prod_{i=0}^{n/2-2}(1-p^{2i-n+2k-2s})(1-p^{2i+1-2s}).$$

Proof. Put $d_0=2^{-\delta_{2,p}n}p^{-l_0}D_0$, and $\tilde{d}_0=p^{-l_0}D_0$. First let $p\neq 2$. Then $l_0=1$ and $D_0=pd_0$. If an element A of $\mathscr{K}_n(\mathbf{Z}_p)$ is type (1,1), A can be expressed as

$$A\sim U_0\perp pU_1,$$

where U_0 (resp. U_1) is a unimodular symmetric matrix of odd degree n_0 (resp. n_1). Put $\Xi=(-1)^{n_1-1}\det U_0$. Then the $GL_n(\mathbf{Z}_p)$-equivalence class of $A\in\mathscr{K}_n(\mathbf{Z}_p)$ is uniquely determined by its determinant d and Ξ. Thus we denote by $A(d;\Xi)$ its representative. Then similarly to Theorem 5.2 we have

$$\begin{aligned}H_p(s;D_0;\omega_p,e)=\sum_{\Xi\in\mathscr{D}}\sum_{l=0}^{n/2-1}\sum_{m=0}^{n/2-l}&p^{(2k-2-n)(l+1/2)}\alpha_p(H_k,A(p^{2l+1}d_0;\Xi))^*\\&\times\alpha_p(A(p^{2l+1}d_0;\Xi),e)p^{(n+1)(l+m+1/2)}p^{-(2l+2m+1)s}\\&\times\omega_p(A(p^{2l+2m+1}d_0;\Xi))\sigma_p(A(p^{2l+2m+1}d_0;\Xi))\\&\times\frac{G_p(A(p^{2l+1}d_0;\Xi),A(p^{2l+2m+1}d_0;\Xi))}{\alpha_p(A(p^{2l+2m+1}d_0;\Xi),A(p^{2l+2m+1}d_0;\Xi))}\\&\times T_p(p^{-2s+2k-2};A(p^{2l+1}d_0;\Xi),A(p^{2l+2m+1}d_0;\Xi))\\&\times B_p(p^{-2s+k-1};A(p^{2l+2m+1}d_0;\Xi)).\end{aligned}$$

By Propositions 4.3, 4.4, and 4.5, the quantity

$$\begin{aligned}
&p^{(2k-2-n)(l+1/2)+(n+1)(l+m+1/2)-(2l+2m+1)s}\\
&\quad\times \alpha_p(H_k, A(p^{2l+1}d_0;\Xi))^*\sigma_p(A(p^{2l+2m+1}d_0;\Xi))\\
&\quad\times \frac{G_p(A(p^{2l+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi))}{\alpha_p(A(p^{2l+2m+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi))}\\
&\quad\times T_p(p^{-2s+2k-2}; A(p^{2l+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi))\\
&\quad\times B_p(p^{-2s+k-1}; A(p^{2l+2m+1}d_0;\Xi))
\end{aligned}$$

is independent of d_0 and Ξ, and it is equal to

$$\Psi_{n,k}p^{-s+k-3/2}(1+p^{-2s+k-1})Q(u^2, v^2, p; n/2-1, l, m),$$

where $u = p^{-s+k-2}, v = p^{-s+n/2-1}$, and

$$\Psi_{n,k} = \frac{(1-p^{-k})\prod_{i=1}^{n/2}(1-p^{-2k+2i})}{2\varphi_{n/2-1}(p^{-2})}.$$

Thus we have

$$\begin{aligned}
H_p(s; D_0; \omega_p, e) &= \Psi_{n,k}p^{-s+k-3/2}(1+p^{-2s+k-1})\\
&\qquad\times \sum\nolimits_{l=0}^{n/2-1}\sum_{m=0}^{n/2-1-l} Q_{-1,0}(u^2, v^2, p; n/2-1, l, m)\\
&\qquad\times \sum_{\Xi\in\mathscr{D}} \omega_p(A(p^{2l+2m+1}d_0;\Xi))\alpha_p(A(p^{2l+1}d_0;\Xi), e).
\end{aligned}$$

If we have $\omega_p = \iota_p$, then by Propositions 4.1 and 4.2 we have

$$\sum_{\Xi\in\mathscr{D}} \iota_p(A(p^{2l+2m+1}d_0;\Xi))\alpha_p(A(p^{2l+1}d_0;\Xi), e) = 2$$

for any l and m. Thus assertion (1) follows from (2.2) of Lemma 4.6.

Let $\omega_p = h_p$. Then by Proposition 4.1 and Lemma 5.1 we have

$$\sum_{\Xi\in\mathscr{D}} h_p(A(p^{2l+2m+1}d_0;\Xi))\alpha_p(A(p^{2l+1}d_0;\Xi), e) = 2p^{1-n/2+l}R_p(e, D_0)$$

for any l and m, and therefore, the assertion follows from (2.1) of Lemma 4.6.

Next let $p = 2$ and for $i = \pm 1$, put

$$\mathscr{W}_{D_0,i} = \{W = c_0 \perp c_1; c_0c_1 = (-1)^{n/2-1}\tilde{d}_0, h_2(W) = i\}/GL_2(\mathbf{Z}_2)$$

or

$$\mathscr{W}_{D_0,i} = \{W = c_0 \perp 2c_1; c_0c_1 = (-1)^{n/2-1}\tilde{d}_0, h_2(W) = i\}/GL_2(\mathbf{Z}_2)$$

according as $v(D_0) = 2$ or 3, and put $\mathscr{W}_{D_0} = \mathscr{W}_{D_0,+1} \cup \mathscr{W}_{D_0,-1}$. We note that the set $\mathscr{W}_{D_0,i}$ consists of a single element for any D_0 and i. Further every element A of $\mathscr{K}_n(\mathbf{Z}_2)^{(1,1)}$ or of $\mathscr{K}_n(\mathbf{Z}_2)^{(2,0)}$ with determinant d is equivalent to the following form:

$$H_{n_0/2} \perp 2H_{n_1/2} \perp W,$$

where $W \in \mathscr{W}_{2^{n_0+2}d}$. This A is uniquely determined by its determinant d and W. Thus we can express this A as $A(d;W)$. Then we have

$$\begin{aligned}
H_2(s;D_0;h_2,e) = 2^{n(s-k+1/2)}& \\
&\times \Bigg[\sum_{W \in \mathscr{W}_{D_0}} \sum_{l=0}^{n/2-1} \sum_{m=0}^{n/2-l} 2^{(2k-2-n)(l+l_0/2)} \alpha_2(H_k, A(2^{2l+l_0}d_0;W))^* \\
&\times \alpha_2(A(2^{2l+l_0}d_0;W),e) 2^{(n+1)(l+m+l_0/2)} 2^{-(2l+2m+l_0)s} \\
&\times \omega_2(A(2^{2l+2m+l_0}d_0;W)) \sigma_2(A(2^{2l+2m+l_0}d_0;W)) \\
&\times \frac{G_2(A(2^{2l+l_0}d_0;W), A(2^{2l+2m+l_0}d_0;W))}{\alpha_2(A(2^{2l+2m+l_0}d_0;W), A(2^{2l+2m+l_0}d_0;W))} \\
&\times T_2(2^{-2s+2k-2}; A(2^{2l+l_0}d_0;W), A(2^{2l+2m+l_0}d_0;W)) \\
&\times B_2(2^{-2s+k-1}; A(2^{2l+2m+l_0}d_0;W)) \Bigg].
\end{aligned}$$

By Propositions 4.3, 4.4, and 4.5, the quantity

$$\begin{aligned}
&2^{(2k-2-n)(l+l_0/2)+(n+1)(l+m+l_0/2)-(2l+2m+l_0)s} \\
&\quad \times \alpha_2(H_k, A(2^{2l+l_0}d_0;W))^* \sigma_2(A(2^{2l+2m+l_0}d_0;W)) \\
&\quad \times \frac{G_2(A(2^{2l+l_0}d_0;W), A(2^{2l+2m+l_0}d_0;W))}{\alpha_2(A(2^{2l+2m+l_0}d_0;W), A(2^{2l+2m+l_0}d_0;W))} \\
&\quad \times T_2(2^{-2s+2k-2}; A(2^{2l+l_0}d_0;W), A(2^{2l+2m+l_0}d_0;W)) \\
&\quad \times B_2(2^{-2s+k-1}; A(2^{2l+2m+l_0}d_0;W)),
\end{aligned}$$

is independent of d_0 and W, and it is equal to

$$\Psi_{n,k} 2^{(-s+k-3/2)l_0}(1 + 2^{-2s+k-1}) Q(u^2, v^2, 2; n/2-1, l, m).$$

Thus we have

$$H_2(s;D_0;\omega_2,e)=2^{n(s-k+1/2)+(-s+k-3/2)l_0}\Psi_{n,k}$$
$$\times\sum_{l=0}^{n/2-1}\sum_{m=0}^{n/2-1}Q_{-1,0}(u^2,v^2,2;n/2-1,l,m)(1+2^{-2s+k-1})$$
$$\times\sum_{W\in\mathscr{W}_{D_0}}\omega_2(A(2^{2l+2m+l_0}d_0;W))\alpha_2(A(2^{2l+l_0}d_0;W),e).$$

On the other hand, by Proposition 4.2 we have

$$\alpha_2(A(2^{2l+l_0}d_0;W),e)=1+h_2((-e)\perp W)2^{(1-n/2)(l_0+r)+l}.$$

By Lemma 5.1 we have

$$h_2(C)=(-1)^{n(n-2)/8}((-1)^{n/2-1},\det W)_2h_2(W).$$

Thus the assertion follows from (2.1) and (2.2) of Lemma 4.6. $\square$

Proof of Theorem 1. Let $D_0\in\mathscr{F}_{(-1)^{n/2}}$. For $\omega_p=\iota_p$ or h_p, put

$$I_p(s;D_0;\omega_p)=\sum_{r=0}^{\infty}b(p^r)p^{r(n/2-k)}H_p(s;D_0;\omega_p;p^r),$$

and

$$I(s;D_0;\{\omega_p\}_p)=\sum_{e=1}^{\infty}b(e)e^{n/2-k}\prod_p H_p(s;D_0;\omega_p;e).$$

We write $e=p^{\nu_p(e)}e_{0,p}$ with $(e_{0,p},p)=1$. First let $\omega_p=\iota_p$. Then $H_p(s;D_0;\iota_p;e)$ depends only on $\nu_p(e)$ and does not depend on $e_{0,p}$. Thus by the multiplicative property of $b(e)$ we have

$$I(s;D_0;\{\iota_p\}_p)=\prod_p I_p(s;D_0;\iota_p).$$

For a prime number p not dividing D_0, put

$$\tilde{I}_p(s;D_0;\iota_p)=I_p(s;D_0;\iota_p)\Bigg\{2^{n\delta_{2,p}(s-k+1/2)}\Upsilon_{n,k}(p)$$
$$\times\prod_{i=0}^{n/2-2}(1-p^{2i-n-1+2k-2s})(1-p^{2i+2-2s})\Bigg\}^{-1},$$

and $\delta = \chi_p((-1)^{n/2}D_0)$. Then by Theorem 5.2 we have

$$\begin{aligned}\tilde{I}_p(s;D_0;\iota_p) &= \left[\sum_{r=0}^{\infty} b(p^r)p^{r(n/2-k)}\left\{p^{-2s+2k-3}(1-p^{n-2k})(1+p^{-k+2})\right.\right.\\ &\quad \left.\left. + \frac{1-(\delta p^{1-n/2})^{r+1}}{1-\delta p^{1-n/2}}(1+\delta p^{n/2-k})(1-p^{-3+2k-2s})(1-p^{-2s})\right\}\right]\\ &= p^{-2s+2k-3}(1-p^{n-2k})(1+p^{-k+2})\zeta_p(f,k-n/2)\\ &\quad + (1-p^{-3+2k-2s})(1-p^{-2s})(1-p^{n-2k})\\ &\quad \times L_p(f,(-1)^{n/2}D_0,k-1)\zeta_p(f,k-n/2),\end{aligned}$$

where $L_p(f,(-1)^{n/2}D_0,*)$ and $\zeta_p(f,*)$ denote the p-Euler factor of $L(f,(-1)^{n/2}D_0,*)$ and $\zeta(f,*)$, respectively. Thus we have

$$\begin{aligned}I_p(s;D_0;\iota_p) &= 2^{n\delta_{2,p}(s-k+1/2)}\Upsilon_{n,k}(p)(1-p^{n-2k})\zeta_p(f,k-n/2)(1-p^{-2s+2k-3})\\ &\quad \times \prod_{i=0}^{n/2-2}(1-p^{-2s+2i+2})(1-p^{-2s+2k+2i-n-1})L_p(f,(-1)^{n/2}D_0,k-1)\\ &\quad \times \{(1+p^{-2s+k-2})(1+p^{-2s+k-1})-\delta b(p)p^{-2s+k-2}(1+p^{2-k})\}.\end{aligned}$$

Let $p|D_0$. Then we have

$$\begin{aligned}I_p(s;D_0;\iota_p) &= 2^{n\delta_{2,p}(s-k+1/2)}\Upsilon_{n,k}(p)(1-p^{n-2k})p^{-s+k-3/2}\zeta_p(f,k-n/2)\\ &\quad \times (1+p^{-2s+k-1})\prod_{i=0}^{n/2-2}(1-p^{-2s+2i+2})(1-p^{-2s+2k+2i-n-1}).\end{aligned}$$

Note that $\delta = \psi_{(-1)^{n/2}D_0}(p)$. Thus we have

$$\begin{aligned}I(s;D_0;\{\iota_p\}_p) &= \frac{2^{ns-nk+n/2}\prod_{i=1}^{n/2-1}\zeta(2i)}{\zeta(k)\prod_{i=1}^{n/2}\zeta(2k-2i)}\\ &\quad \times \frac{D_0^{-s+k-3/2}L(f,(-1)^{n/2}D_0,k-1)\zeta(f,k-n/2)}{\prod_{i=0}^{n/2-2}\zeta(2s-2i-2)\zeta(2s-2k-2i+n+1)}\\ &\quad \times \prod_p\{(1+\psi_{(-1)^{n/2}D_0}(p)^2p^{-2s+k-2})\\ &\quad \times (1+p^{-2s+k-1})-\psi_{(-1)^{n/2}D_0}(p)b(p)p^{-2s+k-2}(1+p^{2-k})\}.\end{aligned}$$

Next let $\omega_p = h_p$, and for a positive integer e put

$$\tilde{H}_p(s, D_0, h_p, e) = H_p(s, D_0, h_p, e)\Bigg\{2^{n\delta_{2,p}(s-k+1/2)}\Upsilon_{n,k}(p) \times \prod_{i=0}^{n/2-2}(1-p^{2i-n+2k-2s})(1-p^{2i+1-2s})\Bigg\}^{-1},$$

$$\tilde{I}_p(s, D_0; h_p) = I_p(s, D_0; h_p)\Bigg\{2^{n\delta_{2,p}(s-k+1/2)}\Upsilon_{n,k}(p) \times \prod_{i=0}^{n/2-2}(1-p^{2i-n+2k-2s})(1-p^{2i+1-2s})\Bigg\}^{-1},$$

and

$$\tilde{I}(s, D_0; \{h_p\}_p) = \prod_p \tilde{I}_p(s, D_0; h_p).$$

Then the situation is more complex than the above case. In fact, if $p|D_0$, then $\tilde{H}_p(s, D_0, h_p, e)$ depends not only on $v_p(e)$ but also on $e_{0,p}$, and it is expressed as

$$\tilde{H}_p(s, D_0, h_p, e) = R_p(e, D_0)(1-p^{n-2k})(1+p^{-2s+k-1}),$$

where $R_p(e, D_0)$ is the one in Theorem 5.3. On the other hand, if $(p, D_0) = 1$, then $\tilde{H}_p(s, D_0, h_p, e)$ does not depend on $e_{0,p}$. Then we have

$$\prod_p \tilde{H}_p(s, D_0, h_p, e) = \prod_{(p,D_0)=1} \tilde{H}_p(s, D_0, h_p, p^{v(e)}) \prod_{p|D_0} \tilde{H}_p(s) \prod_{p|D_0} R_p(e, D_0),$$

where

$$\tilde{H}_p(s) = (1-p^{n-2k})(1+p^{-2s+k-1}).$$

Now we compute $\prod_{p|D_0} R_p(e, D_0)$. Put $D_0 = 2^{m_0}D_0'$ with D_0' odd integer. Let $e = q^2\tilde{e}$ with q a positive integer and $\tilde{e}$ a square free integer. Furthermore, let q_0 be the greatest common divisor of D_0 and $\tilde{e}$, and $e' = \tilde{e}/q_0$. First assume that $m_0 = 0$. Then we have $D_0' = D_0$ and $(-1)^{n/2}D_0 \equiv 1 \bmod 4$. Thus we have

$$\prod_{p|D_0} R_p(e, D_0)\left(\prod_{p|D_0} p^{v_p(e)(1-n/2)}\right)^{-1} = (-1)^{n/2}\left(\frac{e'}{D_0}\right) = (-1)^{n/2}\psi_{(-1)^{n/2}D_0}(e').$$

Next assume that $m_0 = 2$ and $(-1)^{n/2}D_0{}' \equiv 3 \bmod 4$, or $m_0 = 3$. Then in the same manner as above we have

$$\prod_{p|D_0} R_p(e, D_0)\left(\prod_{p|D_0} p^{v_p(e)(1-n/2)}\right)^{-1} = (-1)^{n(n-2)/8}\psi_{(-1)^{n/2}D_0}(e').$$

We note that $\psi_{(-1)^{n/2}D_0}(e') = \psi_{(-1)^{n/2}D_0}(m^2e')$ for any integer m prime to D_0. Thus we have

$$\begin{aligned}
\tilde{I}(s, D_0; \{h_p\}_p) &= (-1)^{n(n-2)/8}D_0^{-s+k-(n+1)/2}\sum_{e=1}^{\infty}\psi_{(-1)^{n/2}D_0}(e')b(e)e^{-k+n/2} \\
&\quad\times \prod_{p|D_0}(\tilde{H}_p(s)p^{v_p(e)(1-n/2)})\prod_{(p,D_0)=1}\tilde{H}_p(s, D_0, h_p, p^{v(e)}) \\
&= (-1)^{n(n-2)/8}D_0^{-s+k-(n+1)/2}\prod_{p|D_0}\tilde{H}_p(s)\sum_{r=0}^{\infty}b(p^r)p^{(-k+1)r} \\
&\quad\times \sum_{(e_0,D_0)=1}\prod_{(p,D_0)=1}\psi_{(-1)^{n/2}D_0}(e_0)b(e_0)\tilde{H}_p(s, D_0, h_p, p^{v(e_0)})e_0^{-k+n/2} \\
&= (-1)^{n(n-2)/8}D_0^{-s+k-(n+1)/2}\prod_{p|D_0}\tilde{H}_p(s)\prod_{p|D_0}\zeta_p(f, k-1) \\
&\quad\times \prod_{(p,D_0)=1}\sum_{r=0}^{\infty}\psi_{(-1)^{n/2}D_0}(p^r)b(p^r)\tilde{H}_p(s, D_0, h_p, p^r)p^{r(-k+n/2)}.
\end{aligned}$$

Now for a prime number p such that $(p, D_0) = 1$ put

$$\tilde{J}_p(s, D_0, h_p) = \sum_{r=0}^{\infty}\psi_{(-1)^{n/2}D_0}(p^r)b(p^r)\tilde{H}_p(s, D_0, h_p, p^r)p^{r(-k+n/2)}.$$

Then, in a manner similarly to the case $\omega_p = \iota_p$, we have

$$\begin{aligned}
\tilde{J}_p(s, D_0, h_p) &= \zeta_p(f, k-1)L_p(f, (-1)^{n/2}D_0, k-n/2)(1-p^{n-2k}) \\
&\quad\times \{(1+p^{-2s+k-1})(1+p^{-2s+k-2}) - \delta b(p)p^{-2s+k-n/2-1}(1+p^{n-k})\}.
\end{aligned}$$

Thus we have

$$I(s; D_0; \{h_p\}_p) = \frac{(-1)^{n(n-2)/8} 2^{ns-nk+n/2} \prod_{i=1}^{n/2-1} \zeta(2i)}{\zeta(k) \prod_{i=1}^{n/2} \zeta(2k-2i)}$$
$$\times \frac{D_0^{-s+k-(n+1)/2} L(f, (-1)^{n/2} D_0, k-n/2) \zeta(f, k-1)}{\prod_{i=0}^{n/2-2} \zeta(2s-2i-1)\zeta(2s-2k-2i+n)}$$
$$\times \prod_p \{(1+\psi_{(-1)^{n/2}D_0}(p)^2 p^{-2s+k-2})(1+p^{-2s+k-1})$$
$$- \psi_{(-1)^{n/2}D_0}(p) b(p) p^{-2s+k-n/2-1}(1+p^{n-k})\}.$$

We note that

$$\zeta^{\mathrm{st}}([f]_1^n; 2s-k+1) = \zeta^{\mathrm{st}}(f, 2s-k+1) \prod_{i=1}^{n-1} \zeta(2s-i)\zeta(2s-2k+i+2).$$

Thus the assertion follows from Theorem 3.2 and the remark before Theorem 1. □

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